1.

Let $P(x,y)$ be the assertion $f(xf(y)+f(y))=f(x)^2+y$
$P(0,x) \Rightarrow f(f(x))=x+f(0)^2$ and so $f(x)$ is bijective
and so $\exists a \in \mathbb{R}$ such that $f(a)=0$.

Then $P(a,x) \Rightarrow f(f(x)=x$ and $f(x)=0$.

$P(f(x),y) \Rightarrow f(f(x)f(f(x))+f(y)=f(f(x))^2+y$,
and so $f(xf(x)+f(y))=x^2+y$
and so, by comparing with $P(x,y)$ we get $f(x)^2=x^2$.
So, $\forall x \in \mathbb{R}$, either $f(x)=x$ or $f(x)=-x$.

Suppose now $\exists a$ such $f(a)=-a$ and $\exists b$ such $f(b)=b$ and $ab \neq 0$.
$P(a,b) \Rightarrow f(-a^2+b)=a^2+b$
and so either $-a^2+b=a^2+b$ or $a^2-b=a^2+b \Rightarrow$ either $a=0$ or $b=,0$ a contradiction.

So, either $f(x)=x \quad \forall x$, or $f(x)=-x \quad \forall x$
It is easy to check that both these solution fit the requirements.
Hence, the two solutions are:
$f(x)=x \quad \forall x \in \mathbb{R}$.
$f(x)=-x \quad \forall x \in \mathbb{R}$.

2.

we set $a+b=x, b+c=y, a+c=z$ and get

$$2f(a,b,c)=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-6$$
$$=\underbrace{\frac{x}{y}+\frac{y}{x}}_{\geq 2}+\underbrace{\frac{x}{z}+\frac{z}{x}}_{\geq 2}+\underbrace{\frac{y}{z}+\frac{z}{y}}_{\geq 2}-3 \geq 3.$$

3.

If A places -1 in front of the term x and at its second move he places an integer in the last free place, which is the opposite of what B placed, then the equation has the form $x^3 - ax^2 - x + a = 0$. This equation has the roots $-1,\ 1,\ a$, which are integers.

4.

Let $z_n = a_n + ib_n$ where $i^2 = -1$

From the definition,

$$z_n = a_0 a_{n-1} - b_0 b_{n-1} + i(a_0 b_{n-1} + b_0 a_{n-1})$$
$$z_n = a_0(a_{n-1} + ib_{n-1}) - b_0 b_{n-1} + ib_0 a_{n-1}$$
$$z_n = a_0(a_{n-1} + ib_{n-1}) + i^2 b_0 b_{n-1} + ib_0 a_{n-1}$$
$$z_n = a_0(a_{n-1} + ib_{n-1}) + ib_0(a_{n-1} + ib_{n-1})$$
$$z_n = (a_0 + ib_0)(a_{n-1} + ib_{n-1})$$

$$\therefore z_n = z_0 z_{n-1}$$

Therefore $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a GP with first term and common ratio z_0. Also, note that $|z_0| < 1$.

$$\therefore \sum_{n=0}^{\infty} a_n + i\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} z_n$$

$$\begin{aligned}\sum_{n=0}^{\infty} z_n &= z_0 + z_1 + z_2 + \dots \\ &= z_0 + z_0^2 + z_0^3 + \dots \\ &= \frac{z_0}{1 - z_0} \\ &= \frac{\frac{1}{2} + i\frac{1}{3}}{\frac{1}{2} - i\frac{1}{3}}\end{aligned}$$

$$\sum_{n=0}^{\infty} a_n + i\sum_{n=0}^{\infty} b_n = \frac{5}{13} + i\frac{12}{13}$$

5.

$$
\begin{aligned}
&x^2y''' + 3xy'' + y' = \frac{1}{1+x} \\
&\implies x^2y''' + 2xy'' + xy'' + y' = \frac{1}{1+x} \\
&\implies (x^2y'')' + (xy')' = \frac{1}{1+x} \\
&\implies x^2y'' + xy' = \log(1+x) + C
\end{aligned}
$$

$$
\begin{aligned}
&\text{Keeping } x = 0 \implies C = 0 \\
&\because x \in [0,1),\ \text{using expansion of } \log(1+x): \\
&x^2y'' + xy' = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \\
&\implies xy'' + y' = 1 - \frac{x}{2} + \frac{x^2}{3} - \dots \\
&\implies (xy')' = 1 - \frac{x}{2} + \frac{x^2}{3} - \dots \\
&\implies xy' = x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \dots + C
\end{aligned}
$$

$$
\begin{aligned}
&\text{Keeping } x = 0 \implies C = 0 \\
&\implies y' = 1 - \frac{x}{2^2} + \frac{x^2}{3^2} - \dots \\
&\implies y = x - \frac{x^2}{2^3} + \frac{x^3}{3^3} + \dots + C \\
&y(0) = 0 \implies C = 0 \\
&\therefore y = f(x) = x - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \dots
\end{aligned}
$$

let us first evaluate the following sum:

$$S = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \dots$$

It is given that:

$$
\begin{aligned}
&1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90} \\
&\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots = \frac{1}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right) \\
&\qquad\qquad\qquad\qquad\quad = \frac{1}{16} \cdot \frac{\pi^4}{90} \\
&(1) - 2 \cdot (2): \\
&1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \dots = \frac{\pi^4}{90}\left(1 - \frac{1}{8}\right) \\
&\therefore S = \frac{7\pi^4}{720}
\end{aligned}
$$

Now,

$$I(n) = \int_0^1 \frac{f(x^n)}{x} dx$$

$$= \int_0^1 \frac{1}{x} \left(x^n - \frac{x^{2n}}{2^3} + \frac{x^{3n}}{3^3} - \dots \right) dx$$

$$= \int_0^1 \left(x^{n-1} - \frac{x^{2n-1}}{2^3} + \frac{x^{3n-1}}{3^3} - \dots \right) dx$$

$$= \frac{1}{n} - \frac{1}{(2n)(2)} + \frac{1}{(3n)(3)} - \dots$$
$$= \frac{1}{n} \left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \dots \right)$$
$$= \frac{S}{n}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{I(k)}{k} = \sum_{k=1}^{\infty} \frac{S}{k^2} = S \cdot \zeta(2)$$

$$= \boxed{\frac{7\pi^6}{4320}}$$

6.

Note:

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)\Bigg|_{x=1} = \frac{d^n}{dx^n}\left(\frac{\log(1+x)}{1+x}\right)\Bigg|_{x=0}$$

$$\frac{\log(1+x)}{1+x} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)\left(1 - x + x^2 - x^3 - \dots\right)$$

$$\text{Co-efficient of } x^n \text{ in } (2) = (-1)^{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}\right)$$

For evaluating the n-th derivative of a polynomial (or more appropriately, power series here) at x=0, only information of the co-efficient of x^n is required as all the lower powers will disappear due to repeated differentiation and higher powers will vanish on putting x=0. (This is why the transformation in (1) was done.)
Formally:

$$\begin{aligned} \text{If } f(x) &= a_0 + a_1 x + a_2 x^2 + \dots \\ &= \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

Then,

$$\frac{d^n}{dx^n} f(x)\Bigg|_{x=0} = n! \cdot a_n \tag{9}$$

Thus, from (3) and (4):

$$g(n) = (-1)^{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n}\right) \tag{10}$$

Now, note that for k $\in \mathbb{N}$, the following holds:

$$g(2k) + g(2k+1) = -\left(1 + \frac{1}{2} + \dots \frac{1}{2k}\right) + \left(1 + \frac{1}{2} + \dots \frac{1}{2k+1}\right) = \frac{1}{2k+1} \tag{11}$$

Finally,

$$\begin{aligned} \sum_{k=2m}^{4m+1} g(k) &= g(2m) + g(2m+1) + g(2m+2) + g(2m+3) + \dots g(4m) + g(4m+1) \\ &= \frac{1}{2m+1} + \frac{1}{2m+3} + \dots \frac{1}{4m+1} \qquad \text{(From (6))} \end{aligned}$$

$$= \sum_{r=0}^{m} \frac{1}{2(m+r)+1}$$

$$\begin{aligned}
\therefore \lim_{m\to\infty} \sum_{k=2m}^{4m+1} g(k) &= \lim_{m\to\infty} \sum_{r=0}^{m} \frac{1}{2(m+r)+1} \\
&= \lim_{m\to\infty} \frac{1}{2}\frac{1}{m} \sum_{r=0}^{m} \frac{1}{1+\frac{r+1/2}{m}}
\end{aligned}$$

Note that $\dfrac{1}{1+x}$ is continuous and bounded in $[0,1]$. Therefore, it is Riemann integrable. By partitioning the interval and choosing the tags suitably, the limit of the sum can be converted into an integral.

$$\begin{aligned}
\therefore \lim_{m\to\infty} \sum_{k=2m}^{4m+1} g(k) &= \frac{1}{2}\int_0^1 \frac{1}{1+x}dx \\
&= \frac{1}{2}\Big[\log(1+x)\Big]_0^1 \\
&= \boxed{\frac{1}{2}\log 2}
\end{aligned}$$

7.

We have that

$$\begin{aligned}
a^2 + b^2 &= c^2 \\
c - b &= 1
\end{aligned}$$

Claim 1. a is odd.

Proof. From (2), we have that b and c are of different parities. (That is, one is odd and one is even.)
From (1), we have that $a^2 = c^2 - b^2 = (c+b)(c-b) = c+b$.
Thus, a^2 is odd as c and b have different parities.
This gives us that a must be odd.

Claim 2. b is divisible by 4.

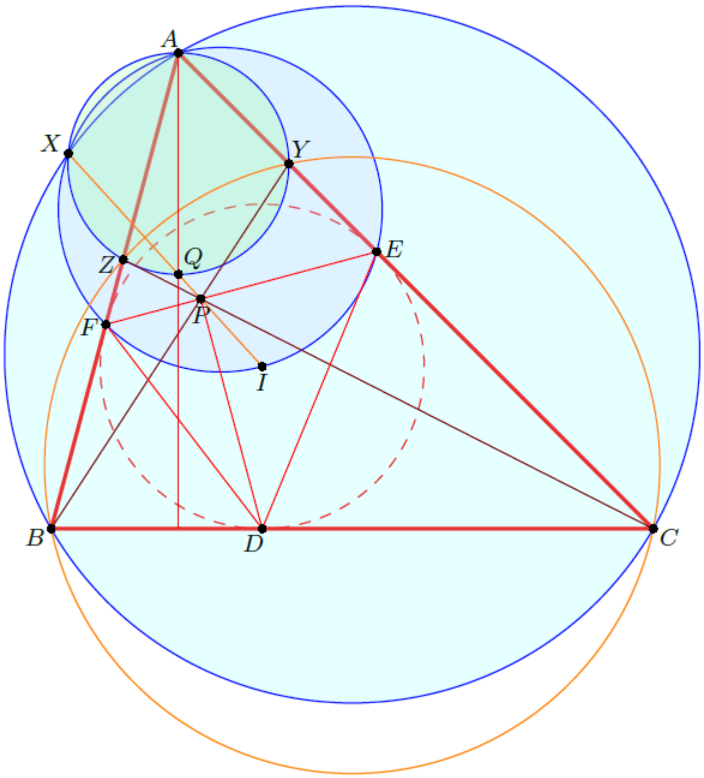
Proof. Plugging the value of c from (2) into (1) gives us:

$$\begin{aligned}
& a^2 + b^2 = (1+b)^2 \\
\implies & 2b = a^2 - 1 \\
\implies & 2b = (2k-1)^2 - 1 && \text{(as } a \text{ is odd, } a = 2k-1 \text{ for some } k \in \mathbb{N}) \\
\implies & b = 2k(k-1)
\end{aligned}$$

As $k(k-1)$ is even, b is divisible by 4.

From (2), we have that $b \equiv -1 \mod c$.
From (1), we have that $a^2 + b^2 \equiv 0 \mod c$.
This means that $b^2 \equiv 1 \equiv -a^2 \mod c$.
That is, $a^2 \equiv -1 \mod c$.
As $b = 4n$ for some $n \in \mathbb{N}$, we have that $a^b = (a^2)^{2n} \equiv (-1)^{2n} \equiv 1 \mod c$.
This gives us that $a^b \equiv 1 \mod c$.
As a is odd and $b \equiv -1 \mod c$, we have that $b^a \equiv -1 \mod c$.
Thus, we have that $a^b + b^a \equiv 0 \mod c$, as desired.

8.



The proof proceeds through a series of seven lemmas.

Lemma 1. Lines DP and EF are the internal and external angle bisectors of $\angle BPC$.

Proof. Since DEF the cevian triangle of ABC with respect to its Gregonne point, we have that

$$-1 = \left(\overline{EF} \cap \overline{BC}, D; B, C\right).$$

Then since $\angle DPF = 90^\circ$ we see P is on the Apollonian circle of BC through D. So the conclusion follows. □

Lemma 2. Triangles BPF and CEP are similar.

Proof. Invoking the angle bisector theorem with the previous lemma gives

$$\frac{BP}{BF} = \frac{BP}{BD} = \frac{CP}{CD} = \frac{CP}{CE}.$$

But $\angle BFP = \angle CEP$, so $\triangle BFP \sim \triangle CEP$. □

Lemma 3. Quadrilateral $BZYC$ is cyclic; in particular, line YZ is the antiparallel of line BC through $\angle BAC$.

Proof. Remark that $\angle YBZ = \angle PBF = \angle ECP = \angle YCZ$. □

Lemma 4. The circumcircles of triangles AYZ, AEF, ABC are concurrent at a point X such that $\triangle XBF \sim \triangle XCE$.

Proof. Note that line EF is the angle bisector of $\angle BPZ = \angle CPY$. Thus

$$\frac{ZF}{FB} = \frac{ZP}{PB} = \frac{YP}{PC} = \frac{YE}{EC}.$$

Then, if we let X be the Miquel point of quadrilateral $ZYCB$, it follows that the spiral similarity mapping segment BZ to segment CY maps E to F; therefore the circumcircle of $\triangle AEF$ must pass through X too. □

Lemma 5. Ray XP bisects $\angle FXE$.

Proof. The assertion amounts to

$$\frac{XF}{XE} = \frac{BF}{EC} = \frac{FP}{PE}.$$

The first equality follows from the spiral similarity $\triangle BFX \sim \triangle CEX$, while the second is from $\triangle BFP \sim \triangle CEP$. So the proof is complete by the converse of angle bisector theorem. □

Lemma 6. Points X, P, I are collinear.

Proof. On one hand, $\angle FXI = \angle FAI = \frac{1}{2}\angle A$. On the other hand, $\angle FXP = \frac{1}{2}\angle FXE = \frac{1}{2}\angle A$. Hence, X, Y, I collinear. □

Lemma 7. Points X, Q, I are collinear.

Proof. On one hand, $\angle AXQ = 90°$, because we established earlier that line YZ was antiparallel to line BC through $\angle A$, hence $AQ \perp BC$ means exactly that $\angle AZQ = AYQ = 90°$. On the other hand, $\angle AXI = 90°$ according to the fact that X lies on the circle with diameter AI. This completes the proof of the lemma. □

Finally, combining the final two lemmas solves the problem.

9.

9. Here, $z_1, z_2, \cdots z_8$ are the vertices of the regular polygon. Let

$$z_n = x + \iota y$$

Now $\frac{1}{a_1 - 2\iota}, \frac{1}{a_2 - 2\iota}, \frac{1}{a_3 - 2\iota}, \frac{1}{a_4 - 2\iota}, \frac{1}{a_5 - 2\iota}, \frac{1}{a_6 - 2\iota}, \frac{1}{a_7 - 2\iota}, \frac{1}{a_8 - 2\iota}$ are also vertices of a regular octagon, where $a_j \in \mathbb{R}$ for $j = 1, 2, \ldots, 8$

So $\frac{1}{a_j - 2\iota} = x + \iota y$ or $\frac{a_j + 2i}{a_j^2 + 4} = x + \iota y$.

This gives us the following:

$$x = \frac{a_j}{a_j^2 + 4}, \quad y = \frac{2}{a_j^2 + 4}.$$

Now, we get that: $x^2 + y^2 = \frac{a_j^2}{(a_j^2 + 4)^2} + \frac{4}{(a_j^2 + 4)^2} = \frac{1}{(a_j^2 + 4)} = \frac{y}{2}$. So, we get that the vertices lie on a circle given by the equation,

$$x^2 + y^2 - \frac{y}{2} = 0.$$

The radius of the circle is $\frac{1}{2}$, and so we have the radius of the circle circumscribing our regular octagon. From the radius, we can easily calculate the area of the octagon.

Area $= \frac{1}{4\sqrt{2}}$

10. Let $N = x_1 x_2 x_3 \cdots x_{10}$ be one such number. Then, we have

$$\sum_{n=1}^{10} x_i = 45.$$

Hence, N is divisible by 9, so N is also divisible by $9 \cdot 11111 = 99999$.

Now, $N = x_1 x_2 x_3 \cdots x_5 .10^5 + x_6 x_7 \cdots x_{10}$

$= x_1 x_2 x_3 \cdots x_5 \cdot 99999 + x_1 x_2 x_3 \cdots x_5 + x_6 x_7 \cdots x_{10}$

Now, $x_1 x_2 x_3 x_3 x_4 x_5 < 99999$

and also, $x_6 x_7 x_8 x_9 x_{10} < 99999$

So, $x_1 x_2 x_3 x_3 x_4 x_5 + x_6 x_7 x_8 x_9 x_{10} < 2 \cdot 99999$.

So, $x_1 x_2 x_3 x_3 x_4 x_5 + x_6 x_7 x_8 x_9 x_{10} = 99999$.

So, $x_1 + x_6 = x_2 + x_7 = x_3 + x_8 = x_4 + x_9 = x_5 + x_{10} = 9$.

So, total number of such numbers is equal to (by solving the number of positive solutions under the conditions of above mentioned equation)

$= 9 \cdot 8 \cdot 6 \cdot 4 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = \mathbf{3456}$.